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Some vorticity theorems and conservation laws for non-barotropic fluids

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Some theorems concerning the vorticity in barotropic flows of perfect fluids are generalized for non-barotropic flows. The generalization involves replacing the velocity in certain parts of the equations by a time-dependent quantity which is a function of the velocity and thermodynamic properties of the fluid. Results which are generalized include Kelvin's circulation theorem and conservation laws for potential vorticity and helicity. It is shown how the results can be further generalized to include dissipative effects. The possibility of using some of the results in deriving a complete set of Lagrangian conservation laws for perfect fluids is discussed.

1. Introduction

There exists a number of theorems and conservation laws concerning the vorticity in barotropic perfect fluids. These include Kelvin's circulation theorem, and conservation laws for potential vorticity and helicity. Eckart (1960) has found a generalized form of Kelvin's circulation theorem which holds for a non-barotropic perfect fluid. It is shown in the present paper that this is a particular case of a generalization which can be applied to several vorticity theorems.

We will use the following equations describing the flow of perfect fluids: the conservation of momentum

$$\frac{D\mathbf{u}}{Dt} = -\nabla(I+\Phi)+T\nabla S, \tag{1}$$

the continuity equation

$$D\rho/Dt+\rho\nabla\,.\,\mathbf{u} = 0, \tag{2}$$

and the conservation of entropy (which implies energy conservation in a perfect fluid)

$$DS/Dt = 0. \tag{3}$$

In the above equations $\mathbf{u}$ is the velocity field, ρ the density, S the specific entropy, I the specific enthalpy and Φ is the potential energy due to any conservative body forces. D/Dt is the material derivative. It is assumed that the equation of state can be written in the form

$$E = E(\rho, S), \tag{4}$$

where E is the specific internal energy. Hence all thermodynamic quantities can be expressed as functions of ρ and S.

It will be necessary to use a generalized form of Weber's transformed equations of motion (Serrin 1959). These can be derived using only equations (1)–(3) and take the form

$$\mathbf{u} = \nabla\phi + \eta\nabla S + \mathbf{u}_0 . \nabla\boldsymbol{\alpha}, \tag{5}$$

$$\frac{D\phi}{Dt} = \tfrac{1}{2}q^2 - I - \Phi, \tag{6}$$

$$\frac{D\eta}{Dt} = T, \quad \frac{D\mathbf{u}_0}{Dt} = 0, \quad \frac{D\boldsymbol{\alpha}}{Dt} = 0 \qquad (7)\text{–}(9)$$

together with equation (3). Equations (6) and (7) define ϕ and η respectively, while $\boldsymbol{\alpha}$ are the Lagrange co-ordinates, $\mathbf{u}_0$ is the initial velocity field of fluid particles and $q^2 = \mathbf{u}.\mathbf{u}$.

We can now define a barotropic flow as one in which $\nabla\eta \times \nabla S = 0$. Although in many flows this is equivalent to the usual requirement that $\nabla T \times \nabla S = 0$, it is not clear whether this is always so. However, it will be shown later that $\nabla\eta \times \nabla S = 0$ is in fact the most general form of the constraint which is necessary in order that many vorticity theorems such as Kelvin's circulation theorem should hold.

The generalization applied by Eckart to Kelvin's circulation theorem involves replacing $\mathbf{u}$ in the result for barotropic fluids by $(\mathbf{u} - \eta\nabla S)$. When applied to other vorticity theorems it is shown in this paper that $\mathbf{u}$ is replaced by $(\mathbf{u} - \eta\nabla S)$ in some but not all of its occurrences in the relevant equations. If, instead, $\mathbf{u}$ is replaced by $\mathbf{u}_0 . \nabla\boldsymbol{\alpha}$, then the theorems still hold. In the case of perfect fluids this generalization is equivalent to the first one, but the results now apply to fluids with dissipation.

One of the theorems concerning the conservation of a generalized potential vorticity forms the basis of one or more of the conservation laws derived by Hollmann (1964) in an attempt to replace the complete set of thermodynamic and hydrodynamic equations for non-barotropic perfect fluids by five Lagrangian conservation laws. Some possible forms of this generalized potential vorticity are discussed with the view to obtaining a complete set of conservation laws. However, it is found that five independent conservation laws do not necessarily completely determine a flow.

2. Generalization of Kelvin's circulation theorem

Eckart (1960) obtained the following generalization of Kelvin's circulation theorem, using a variational principal in Lagrangian co-ordinates:

$$\frac{D}{Dt}\oint_C (\mathbf{u} - \eta\nabla S).d\mathbf{l} = 0; \tag{10}$$

C is a closed curve moving with the fluid but it need not be on an isentropic surface. If it is on such a surface, then equation (2) reduces to the usual result

$$\frac{D}{Dt}\oint_C \mathbf{u}.d\mathbf{l} = 0. \tag{11}$$

In order to obtain equation (2) by more conventional means we begin with the identity (see Batchelor 1970 for a proof)

$$\frac{D}{Dt}\oint_C (\mathbf{u} - \eta\nabla S).d\mathbf{l} = \oint_C \left(\frac{D\mathbf{u}}{Dt} - \frac{D\eta}{Dt}\nabla S - \eta\frac{D(\nabla S)}{Dt}\right).d\mathbf{l} + \oint_C (\mathbf{u} - \eta\nabla S).(d\mathbf{l}.\nabla)\mathbf{u}. \tag{12}$$

From the equation for entropy conservation (equation (3)) we have

$$\frac{D}{Dt}(\nabla S) = -(\nabla S.\nabla)\mathbf{u} - \nabla S \times \boldsymbol{\omega}, \tag{13}$$

where $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ is the vorticity. Using the equations of motion (1), equations (7) and (13), and the identity $\mathbf{u}.(d\mathbf{l}.\nabla)\mathbf{u} = \frac{1}{2}\nabla(q^2).d\mathbf{l}$, it then follows from equation (12) that

$$\frac{D}{Dt}\oint(\mathbf{u} - \eta\nabla S).d\mathbf{l} = -\oint_C [\nabla(I+\Phi) - \eta(\nabla S.\nabla)\mathbf{u} - \eta\nabla S \times \boldsymbol{\omega} - \tfrac{1}{2}\nabla(q^2)].d\mathbf{l} - \oint_C \eta\nabla S.(d\mathbf{l}.\nabla)\,\mathbf{u}. \tag{14}$$

On applying the vector identity

$$-\nabla S.(d\mathbf{l}.\nabla)\mathbf{u} + (\nabla S.\nabla)\mathbf{u}.d\mathbf{l} = -(\nabla S \times \boldsymbol{\omega}).d\mathbf{l}, \tag{15}$$

we can use Stokes' theorem to get the final result

$$\frac{D}{Dt}\oint(\mathbf{u} - \eta\nabla S).d\mathbf{l} = -\oint_C \nabla(I + \Phi - \tfrac{1}{2}q^2).d\mathbf{l} = 0. \tag{16}$$

Starting with equation (16) and using the identities (13) and (15), it is easy to show that

$$\frac{D}{Dt}\oint_C \mathbf{u}.d\mathbf{l} = \oint_C T\,dS, \tag{17}$$

which is Bjerknes' theorem.

3. Generalizations of the Helmholtz theorems

The Helmholtz theorems concerning vortex tubes in barotropic perfect fluids are the following:

(i) If C_1 and C_2 are any two circuits encircling a vortex tube in the same direction, then the circulation around C_1 $\left(\oint_{C_1} \mathbf{u}.d\mathbf{l}\right)$ is equal to the circulation around C_2.

(ii) Vortex lines are material lines.

(iii) The strength of a vortex tube, defined as the circulation around any circuit encircling the tube, remains constant as the tube moves with the fluid.

In a non-barotropic fluid none of these theorems hold so the concept of the strength of a vortex tube is not a useful one. However, it turns out that tubes of $(\boldsymbol{\omega} - \nabla\eta \times \nabla S)$ have all the properties of vortex tubes in barotropic fluids.

Consider a segment of such a tube (figure 1). The circuits C_1 and C_2 encircling the tube enclose areas Σ_1 and Σ_2. Using the vector identity

$$\nabla.(\boldsymbol{\omega} - \nabla\eta \times \nabla S) \equiv 0,$$

we have

$$\int_V \nabla.(\boldsymbol{\omega} - \nabla\eta \times \nabla S)\,dV = 0, \tag{18}$$

where V is the volume of the segment. Then, because $(\boldsymbol{\omega} - \nabla\eta \times \nabla S)$ is parallel to the surface of the volume everywhere on the surface except on its ends, the divergence theorem gives

$$\int_{\Sigma_1}(\boldsymbol{\omega} - \nabla\eta \times \nabla S).d\boldsymbol{\Sigma}_1 - \int_{\Sigma_2}(\boldsymbol{\omega} - \nabla\eta \times \nabla S).d\boldsymbol{\Sigma}_2 = 0. \tag{19}$$

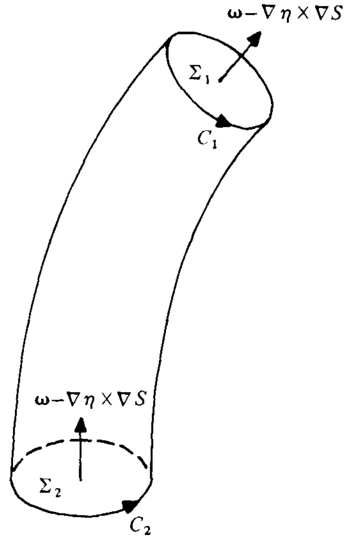

FIGURE 1. A segment of a tube of $(\boldsymbol{\omega}-\nabla\eta\times\nabla S)$.

Then by Stokes' theorem

$$\oint_{C_1}(\mathbf{u}-\eta\nabla S).d\mathbf{l}_1=\oint_{C_2}(\mathbf{u}-\eta\nabla S).d\mathbf{l}_2, \tag{20}$$

i.e. the circulations around C_1 and C_2 are equal. This is a generalization of the first Helmholtz theorem.

Consider a closed circuit C lying on a surface composed entirely of lines of

$$(\boldsymbol{\omega}-\nabla\eta\times\nabla S).$$

The generalized circulation $\oint_C(\mathbf{u}-\eta\nabla S).d\mathbf{l}$ is then zero. If the surface moves with the fluid carrying the line C with it then the generalized circulation will always be zero by the generalized circulation theorem. Therefore the surface will always consist of lines of $(\boldsymbol{\omega}-\nabla\eta\times\nabla S)$. Two such moving surfaces must intersect along a line of $(\boldsymbol{\omega}-\nabla\eta\times\nabla S)$ so these lines are material lines. This is the generalization of the second Helmholtz theorem and the proof is essentially that given by Lamb (1932) for the barotropic case.

From the second generalized Helmholtz theorem and the generalized circulation theorem it follows that the strength of a tube of $(\boldsymbol{\omega}-\nabla\eta\times\nabla S)$ is constant in time. This generalizes the third Helmholtz theorem.

4. An alternative form of the vorticity equation

For a barotropic fluid the vorticity equation can be written in the form

$$\frac{D}{Dt}\left(\frac{\boldsymbol{\omega}}{\rho}\right)=\frac{\boldsymbol{\omega}}{\rho}.\nabla\mathbf{u}, \tag{21}$$

where ρ is the density. For a non-barotropic fluid this generalizes to

$$\frac{D}{Dt}\left(\frac{\boldsymbol{\omega}-\nabla\eta\times\nabla S}{\rho}\right)=\left(\frac{\boldsymbol{\omega}-\nabla\eta\times\nabla S}{\rho}\right).\nabla\mathbf{u}. \tag{22}$$

To prove this we use the result

$$\frac{D}{Dt}\left(\frac{\nabla\eta \times \nabla S}{\rho}\right) = \left(\frac{\nabla\eta \times \nabla S}{\rho}\right).\nabla\mathbf{u} + \frac{\nabla T \times \nabla S}{\rho}. \tag{23}$$

This follows by expanding the term on the left-hand side of (23) and using equation (7), the continuity equation (2) and the vector identity

$$\nabla S \times (\nabla\eta.\nabla)\mathbf{u} - \nabla\eta \times (\nabla S.\nabla)\mathbf{u} + \nabla S \times (\nabla\eta \times \boldsymbol{\omega}) - \nabla\eta \times (\nabla S \times \boldsymbol{\omega}) = (\nabla S \times \nabla\eta)\nabla.\mathbf{u} - (\nabla S \times \nabla\eta).\nabla\mathbf{u}. \tag{24}$$

Vazsonyi's vorticity equation for a non-barotropic fluid is (see, for example, Serrin 1959)

$$\frac{D}{Dt}\left(\frac{\boldsymbol{\omega}}{\rho}\right) = \frac{\boldsymbol{\omega}}{\rho}.\nabla\mathbf{u} + \frac{\nabla T \times \nabla S}{\rho}. \tag{25}$$

Subtracting (23) from (25) gives the required result (22).

Equation (21) integrates immediately (Serrin 1959) to give

$$\frac{\boldsymbol{\omega}}{\rho} = \frac{\boldsymbol{\omega}_0}{\rho_0}.\mathrm{Grad}\,\mathbf{x}, \tag{26}$$

where $\boldsymbol{\omega}_0$ and ρ_0 are the initial vorticity and density of fluid particles respectively, $\mathbf{x}$ is the position of a fluid particle and Grad $\equiv \partial/\partial\alpha_i$, α_i $(i = 1, 2, 3)$ being Lagrange coordinates. Equation (26) generalizes for a non-barotropic fluid to

$$\frac{\boldsymbol{\omega} - \nabla\eta \times \nabla S}{\rho} = \frac{\boldsymbol{\omega}_0}{\rho_0}.\mathrm{Grad}\,\mathbf{x}, \tag{27}$$

if η is measured by integrating the temperature along trajectories from time $t = 0$.

5. A generalized form of helicity which is conserved

It has been shown by Moffatt (1969) that for a barotropic fluid the helicity, H, defined by

$$H = \int \mathbf{u}.\boldsymbol{\omega}\,dV,$$

the integral being over the whole volume of the fluid, is an invariant if $\boldsymbol{\omega}$ is everywhere parallel to the boundaries of the volume. Physically, H is non-zero only when the vortex lines are knotted.

For a non-barotropic fluid H is no longer constant. Consider

$$\frac{D}{Dt}\left(\frac{\mathbf{u}.\boldsymbol{\omega}}{\rho}\right) = \frac{D\mathbf{u}}{Dt}.\frac{\boldsymbol{\omega}}{\rho} + \mathbf{u}.\frac{D}{Dt}\left(\frac{\boldsymbol{\omega}}{\rho}\right).$$

With use of equations (1) and (25) this becomes

$$\frac{D}{Dt}\left(\frac{\mathbf{u}.\boldsymbol{\omega}}{\rho}\right) = \frac{T\nabla S.\boldsymbol{\omega}}{\rho} - \frac{\nabla(I+\Phi).\boldsymbol{\omega}}{\rho} + \mathbf{u}.\left(\frac{\boldsymbol{\omega}}{\rho}.\nabla\right)\mathbf{u} + \mathbf{u}.\frac{\nabla T \times \nabla S}{\rho}. \tag{28}$$

By means of the identity

$$\frac{D}{Dt}\int X\,dV = \int \rho\frac{D}{Dt}\left(\frac{X}{\rho}\right)dV \tag{29}$$

(see Batchelor 1970 for a proof)

for any scalar X, we have after a little manipulation

$$\frac{dH}{dt} = \int (\tfrac{1}{2}q^2 - I - \Phi)\,\boldsymbol{\omega}\,.\,\hat{\mathbf{n}}\,d\Sigma + \int T\nabla S\,.\,\boldsymbol{\omega}\,dV + \int \mathbf{u}\,.\,\nabla T \times \nabla S\,dV, \tag{30}$$

where Σ is the surface of the volume V and $\hat{\mathbf{n}}$ is the unit vector normal to the surface. Clearly H can change even if $\boldsymbol{\omega}\,.\,\hat{\mathbf{n}} = 0$ on the boundaries.

A generalized helicity which is constant in a non-barotropic fluid under appropriate conditions is $(\mathbf{u} - \eta\nabla S)\,.\,(\boldsymbol{\omega} - \nabla\eta \times \nabla S)$. To see this we consider

$$\frac{D}{Dt}\left(\frac{(\mathbf{u} - \eta\nabla S)\,.\,(\boldsymbol{\omega} - \nabla\eta \times \nabla S)}{\rho}\right) = \frac{D}{Dt}\left(\frac{\mathbf{u}\,.\,\boldsymbol{\omega}}{\rho}\right) - \frac{D}{Dt}\left(\mathbf{u}\,.\,\frac{\nabla\eta \times \nabla S}{\rho}\right) - \frac{D}{Dt}\left(\eta\frac{\boldsymbol{\omega}\,.\,\nabla S}{\rho}\right). \tag{31}$$

The first term on the right-hand side has already been evaluated. The second term is

$$\frac{D}{Dt}\left(\mathbf{u}\,.\,\frac{\nabla\eta \times \nabla S}{\rho}\right) = \frac{D\mathbf{u}}{Dt}\,.\,\frac{\nabla\eta \times \nabla S}{\rho} + \mathbf{u}\,.\,\frac{D}{Dt}\left(\frac{\nabla\eta \times \nabla S}{\rho}\right),$$

which can be simplified using equations (1) and (23). The third term simplifies to

$$\frac{D}{Dt}\left(\eta\frac{\boldsymbol{\omega}\,.\,\nabla S}{\rho}\right) = T\frac{\boldsymbol{\omega}\,.\,\nabla S}{\rho}$$

if Ertel's potential vorticity theorem

$$\frac{D}{Dt}\left(\frac{\boldsymbol{\omega}\,.\,\nabla S}{\rho}\right) = 0$$

is used. After using the identity (29), equation (31) gives

$$\frac{D}{Dt}\int (\mathbf{u} - \eta\nabla S)\,.\,(\boldsymbol{\omega} - \nabla\eta \times \nabla S)\,dV = \int (\tfrac{1}{2}q^2 - I - \Phi)\,(\boldsymbol{\omega} - \nabla\eta \times \nabla S)\,.\,\hat{\mathbf{n}}\,d\Sigma, \tag{32}$$

so the generalized helicity is invariant if $(\boldsymbol{\omega} - \nabla\eta \times \nabla S)\,.\,\hat{\mathbf{n}}$ is zero on the boundaries of the volume V.

It is easy to show by considering discrete tubes of $(\boldsymbol{\omega} - \nabla\eta \times \nabla S)$ that the generalized helicity is non-zero only if the lines of $(\boldsymbol{\omega} - \nabla\eta \times \nabla S)$ are knotted. The proof for the barotropic case is given by Moffatt (1969) and the generalization to the non-barotropic case is straightforward.

6. Generalization of potential vorticity

Let λ be a fluid property satisfying

$$D\lambda/Dt = 0. \tag{33}$$

Then for a barotropic fluid, or if λ is an equilibrium thermodynamic function, i.e. $\lambda = \lambda(S, T)$,

$$\frac{D}{Dt}\left(\frac{\boldsymbol{\omega}\,.\,\nabla\lambda}{\rho}\right) = 0. \tag{34}$$

The quantity $\boldsymbol{\omega}\,.\,\nabla\lambda/\rho$ is the potential vorticity. If λ is entropy, equation (34) is Ertel's theorem. We will consider here a system rotating with constant angular velocity $\boldsymbol{\Omega}$, in which case (34) becomes

$$\frac{D}{Dt}\left(\frac{(\boldsymbol{\omega} + 2\boldsymbol{\Omega})\,.\,\nabla\lambda}{\rho}\right) = 0. \tag{35}$$

For equation (35) to hold for a non-barotropic fluid λ must be a function of S and T only. In the rotating co-ordinate system equation (25) becomes

$$\frac{D}{Dt}\left(\frac{\boldsymbol{\omega}+2\boldsymbol{\Omega}}{\rho}\right) = \left(\frac{\boldsymbol{\omega}+2\boldsymbol{\Omega}}{\rho}\right).\nabla\mathbf{u}+\frac{\nabla T\times\nabla S}{\rho}, \tag{36}$$

and equation (23) is unchanged, so, subtracting it from equation (36), we have

$$\frac{D}{Dt}\left(\frac{\boldsymbol{\omega}+2\boldsymbol{\Omega}-\nabla\eta\times\nabla S}{\rho}\right) = \left(\frac{\boldsymbol{\omega}+2\boldsymbol{\Omega}-\nabla\eta\times\nabla S}{\rho}\right).\nabla\mathbf{u}. \tag{37}$$

For any λ satisfying (33) we have the identity

$$\left(\frac{\boldsymbol{\omega}+2\boldsymbol{\Omega}-\nabla\eta\times\nabla S}{\rho}\right).\frac{D}{Dt}(\nabla\lambda) = -\nabla\lambda.\left[\left(\frac{\boldsymbol{\omega}+2\boldsymbol{\Omega}-\nabla\eta\times\nabla S}{\rho}\right).\nabla\mathbf{u}\right]. \tag{38}$$

Using (37) and (38), we have finally

$$\frac{D}{Dt}\left[\left(\frac{\boldsymbol{\omega}+2\boldsymbol{\Omega}-\nabla\eta\times\nabla S}{\rho}\right).\nabla\lambda\right] = 0, \tag{39}$$

expressing the conservation of generalized potential vorticity.

7. Extension of vorticity theorems to fluids with dissipation

The results of §§ 2–6 relate to inviscid fluids; analogous results for viscous, thermally conducting fluids may be derived by using the generalization of Weber's transformed equations of motion to non-barotropic perfect fluids (see Serrin 1959). This is

$$\operatorname{Grad}\mathbf{x}.\mathbf{u}-\mathbf{u}_0 = \eta\operatorname{Grad}S+\operatorname{Grad}\phi. \tag{40}$$

The quantities $\mathbf{u}_0$, ϕ and η are defined in the introduction. Multiplying equation (40) by Grad $\boldsymbol{\alpha}$ gives equation (5), i.e.

$$\mathbf{u} = \nabla\phi+\eta\nabla S+\mathbf{u}_0.\nabla\boldsymbol{\alpha},$$

which can also be obtained as an Euler–Lagrange equation of an Eulerian variational principle for perfect fluids. From (5) we have

$$\boldsymbol{\omega}-\nabla\eta\times\nabla S = \nabla\mathbf{u}_0\times\nabla\boldsymbol{\alpha}. \tag{41}$$

So for perfect fluids we obtain the following:

(i) Generalized Kelvin's circulation theorem

$$\frac{D}{Dt}\oint(\nabla\mathbf{u}_0\times\nabla\boldsymbol{\alpha}).d\boldsymbol{\Sigma} = \frac{D}{Dt}\oint_C(\mathbf{u}_0.\nabla\boldsymbol{\alpha}).d\mathbf{l} = 0, \tag{42}$$

where the surface $\boldsymbol{\Sigma}$ spans the closed curve C which moves with the fluid.

(ii) Generalized 'vorticity' equation

$$\frac{D}{Dt}\left(\frac{\nabla\mathbf{u}_0\times\nabla\boldsymbol{\alpha}}{\rho}\right) = \left(\frac{\nabla\mathbf{u}_0\times\nabla\boldsymbol{\alpha}}{\rho}\right).\nabla\mathbf{u}. \tag{43}$$

(iii) Conservation of generalized potential vorticity

$$\frac{D}{Dt}\left(\frac{(\nabla\mathbf{u}_0\times\nabla\boldsymbol{\alpha}).\nabla\lambda}{\rho}\right) = 0, \tag{44}$$

for any λ satisfying $D\lambda/Dt = 0$. This result is for inertial co-ordinates; the extension to include rotating co-ordinates is straightforward.

(iv) Conservation of generalized helicity

$$\frac{D}{Dt}\int (\nabla \mathbf{u}_0 \times \nabla \boldsymbol{\alpha}).(\mathbf{u}_0.\nabla \boldsymbol{\alpha})\,dV = 0, \tag{45}$$

if $(\nabla \mathbf{u}_0 \times \nabla \boldsymbol{\alpha}).\hat{\mathbf{n}}$ is zero on the boundaries of V. This is not exactly the same as (32) because we have used $\mathbf{u}_0.\nabla\boldsymbol{\alpha}$ instead of $\nabla\phi + \mathbf{u}_0.\nabla\boldsymbol{\alpha}$. However the potential part of $\mathbf{u}$ makes no contribution to the helicity (see Moffatt 1969).

All the above results can be proved directly using only equations (8) and (9), i.e.

$$\frac{D\mathbf{u}_0}{Dt} = \frac{D\boldsymbol{\alpha}}{Dt} = 0,$$

the continuity equation and several vector identities. Equations (8) and (9) hold even if viscous dissipation and heat conduction are present, so the above results (i)–(iv) still hold in dissipative fluids. However, they are unlikely to be as useful as the results for non-dissipative fluids.

8. Possibility of a complete set of conservation laws for perfect fluids

Hollmann (1964) has attempted to derive a complete set of Lagrangian conservation laws to replace the five primitive equations for perfect fluids, i.e. three momentum equations, energy equation and continuity equation. We can express such a set of laws in the form

$$D\psi_i/Dt = 0, \quad i = 1, \ldots, 5. \tag{46}$$

In an appendix Hollmann suggests the use of equations like (44) as one or more of such a set. An equation expressing the conservation of a generalized helicity density is also derived. However, it is by no means certain that five such conservation laws can entirely replace the usual equations, even if they are all independent. Often they will introduce extra variables needing extra equations to determine them. An example of the use of five particular conservation laws given by Hollmann falls into this category. The five conserved quantities include potential temperature and Ertel's potential vorticity. However, another variable, namely the ϕ introduced in equation (6), enters the equations as an extra variable and so a sixth equation is needed to completely determine the flow. This sixth equation is not a Lagrangian conservation law.

A similar procedure to Hollmann's employs the generalized potential vorticity introduced in the present paper in four of the conserved quantities. Thus we could take

$$\left.\begin{aligned} \psi_j &= \frac{(\nabla \mathbf{u}_0 \times \nabla \boldsymbol{\alpha}).\nabla \alpha_j}{\rho}, \quad j = 1,2,3; \\ \psi_4 &= S, \\ \psi_5 &= \frac{(\nabla \mathbf{u}_0 \times \nabla \boldsymbol{\alpha}).\nabla S}{\rho}, \end{aligned}\right\} \tag{47}$$

with

$$D\psi_j/Dt = 0, \quad i = 1, \ldots, 5. \tag{48}$$

These conserved quantities can be shown to be independent and are simpler than Hollmann's but, like his, they fail to determine the flow completely. A possible method of using them in a single step of a numerical scheme might be as follows.

(i) Determine ψ_i, $i = 1, \ldots, 5$, at time Δt (Δt small) from the initial conditions at time $t = 0$.

(ii) Use ψ_1, ψ_2 and ψ_3 to determine the components of $\nabla \mathbf{u}_0 \times \nabla \boldsymbol{\alpha}$.

(iii) Using $\nabla \mathbf{u}_0 \times \nabla \boldsymbol{\alpha}$ and S, determine ρ from ψ_5.

The thermodynamic state of the fluid is thus determined but not the velocity field. As in Hollmann's case we need further equations to determine $\mathbf{u}$; in this example two more are needed and these are equations (6) and (7), determining ϕ and η respectively. These equations are not Lagrangian conservation laws. Having found ϕ and η, $\mathbf{u}$ could be evaluated without further integration using a generalization of the Weber transformation (5) given by Hollmann.

However, it remains possible that, with suitable choices of λ in equation (44), a complete set of Lagrangian conservation laws could be found.

9. Conclusions

A number of theorems involving the vorticity in a barotropic perfect fluid have been extended so as to apply to non-barotropic fluids. This is achieved by replacing the velocity $\mathbf{u}$ by $(\mathbf{u} - \eta \nabla S)$ in some, but not all, of its occurrences in the relevant equations. It has been shown that the Weber transformation can lead to forms of these generalizations which are still valid when dissipation is present. The possibility arises of using a form of generalized potential vorticity in one or more of a complete set of conservation laws for perfect non-barotropic fluids. However, no such complete set has yet been found.

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REFERENCES

Batchelor, G. K. 1970 *An Introduction to Fluid Dynamics*. Cambridge University Press.

Eckart, C. 1960 *Phys. Fluids* **3**, 421.

Hollmann, G. 1964 *Arch. Met. Geophys. Bioklim.* A **14**, 1.

Lamb, H. 1932 *Hydrodynamics*, § 146. Cambridge University Press.

Moffatt, H. K. 1969 *J. Fluid Mech.* **35**, 117.

Serrin, J. 1959 *Handbuch der Physik*, vol. 8, pp. 125–263. Springer.